The interplay of supersymmetry and $\mathcal{PT}$ symmetry in quantum mechanics: a case study for the Scarf II potential

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Abstract
Motivated by the duality of normalizable states and the presence of the quasi-parity quantum number $q = \pm 1$ in $\mathcal{PT}$ symmetric (non-Hermitian) quantum mechanical potential models, the relation of $\mathcal{PT}$ symmetry and supersymmetry (SUSY) is studied. As an illustrative example, the $\mathcal{PT}$ invariant version of the Scarf II potential is presented, and it is shown that the 'bosonic' Hamiltonian has two *different* 'fermionic' SUSY partner Hamiltonians (potentials) generated from the ground-state solutions with $q = 1$ and $q = -1$. It is shown that the 'fermionic' potentials cease to be $\mathcal{PT}$ invariant when the $\mathcal{PT}$ symmetry of the 'bosonic' potential is spontaneously broken. A modified $\mathcal{PT}$ symmetry inspired SUSY construction is also discussed, in which the SUSY charge operators contain the antilinear operator $\mathcal{T}$. It is shown that in this scheme the 'fermionic' Hamiltonians are just the complex conjugates of the original 'fermionic' Hamiltonians, and thus possess the same energy eigenvalues.

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1. Introduction

Symmetries and invariance properties are among the most characteristic features of any physical system. They usually give a deeper insight into the physical nature of the problem, but also help their mathematical formulation. Symmetries typically lead to characteristic patterns in the energy spectrum of the system. These features are shared by the 'classic' potential

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problems of non-relativistic quantum mechanics. Technically, these are relatively simple systems, and accordingly they include a number of exactly solvable examples; nevertheless, they represent the showcase of a wide variety of symmetry and invariance concepts. The most widely known symmetries of quantum mechanical potentials are based on group theory (in particular, Lie algebras), supersymmetry and $\mathcal{PT}$ symmetry.

Group theoretical approaches to quantum mechanical problems and potentials in particular are practically as old as quantum mechanics itself. The elements of the (symmetry, spectrum generating, dynamical [1] and potential [2]) algebras typically connect different eigenstates of the same Hamiltonian or some interrelated Hamiltonians, while the states themselves belong to the irreducible representations of the corresponding group.

A less immediate application of the concept of symmetry appears in supersymmetric quantum mechanics (SUSYQM) [3], where the supersymmetry relates two Hamiltonians which typically have identical spectra except, possibly, the ground state of one of the Hamiltonians which is missing from the spectrum of the other one. For traditional reasons these two Hamiltonians are called the 'bosonic' and the 'fermionic' Hamiltonians and are denoted by the '−' and '+' indices. In SUSYQM they are constructed from two linear first-order differential equations as $H_- = A^\dagger A$ and $H_+ = AA^\dagger$. SUSYQM is essentially a reformulation of the factorization technique which is an old method of generating isospectral potentials [4].

The most recent symmetry concept is the so-called $\mathcal{PT}$ symmetry of one-dimensional quantum mechanical potentials. It has several relevant and interesting implications regarding the energy spectrum. In $\mathcal{PT}$ symmetric quantum mechanics of Bender and Boettcher [5], the potentials are invariant under the simultaneous action of the space and time reflection operations $\mathcal{P}$ and $\mathcal{T}$, and have the property $[V(-x)]^* = V(x)$. A peculiar feature of these models is that although they are not Hermitian, they may possess real bound-state energy spectra. Alternatively, the $\mathcal{PT}$ symmetric potentials may support eigenvalues arranged into complex conjugate pairs [6], but then the energy eigenfunctions cease to be eigenfunctions of the $\mathcal{PT}$ operator, and the emergence of a complex energy can be interpreted as a manifestation of the spontaneous breakdown of $\mathcal{PT}$ symmetry.

In the above symmetry-based scenario occurring in numerous applications of quantum mechanics, an interplay may be noticed between different symmetry concepts. For example, the practical identity of the supersymmetric shift operators A and $A^\dagger$ with the ladder operators of some potential algebras has been established [7] in the case of potential families corresponding to type A and B factorizations [4]. Similarly, in spite of the comparative novelty of the $\mathcal{PT}$ symmetric quantum mechanics, some standard Lie-algebraic methods have already found their inspiring applications within its non-standard framework. Some solvable $\mathcal{PT}$ symmetric potentials have been associated with the $sl(2, C)$ [8, 9], $su(1, 1) \simeq so(2, 1)$ [10] and $so(2, 2)$ [11] potential algebras.

The relationship between the supersymmetric and $\mathcal{PT}$ symmetric considerations may still be felt as a certain 'missing link'. This motivated our forthcoming analysis. Firstly, on the background of [12] we imagined that there exist essential differences between any Hermitian and non-Hermitian versions of the supersymmetric formalism. We believe that it deserves a deeper study, first of all, via particular examples. Secondly, we were always aware that more attention has to be paid to one of the most specific features of the $\mathcal{PT}$ symmetric Hamiltonians, namely, the existence of the so-called quasi-parity quantum number q which, roughly speaking, reflects the emergence of new normalizable states during the transition from the Hermitian to non-Hermitian H. This extension of the basis states has already been investigated in an algebraic formalism [11]. Last but not least, we were encouraged by the increasing number of available $\mathcal{PT}$ symmetric examples where the reality of spectra was explained using techniques of SUSYQM [13–17].

In our paper we shall pay more attention to the recent observation [18] that the quasi-parity may play a key role in the latter context. We are going to emphasize that in a way which extends the scope of the latter reference, the formal changes of the SUSYQM rules in the non-Hermitian case are *not* an artefact of the presence of the singularities in the complex plane of x. For the sake of clarity of our argument, we shall first pick up one characteristic potential (often called Scarf II) and summarize its known properties in section 2. Section 3 is then devoted to its deeper analysis. We shall see that the supersymmetrization of this potential in its fully regular $\mathcal{PT}$ symmetric version exhibits some properties which make it very different from its standard treatment within the Hermitian SUSYQM. The discussion and summary of our findings are finally provided in sections 4 and 5, respectively.

2. The Scarf II potential and its $\mathcal{PT}$ symmetric version

In a way which reflects the innovative character of the $\mathcal{PT}$ symmetric models, many of their studies focused on a particular potential. The first examples of these potentials have even been found using perturbation techniques [19]. Further ones have been identified using semiclassical approximations [5, 20] and numerical algorithms [21]. A number of the exactly solvable $\mathcal{PT}$ symmetric potentials have been revealed as the analogues of their Hermitian, real special cases [13, 15, 22–24]. In such a setting we shall pick up the Scarf II potential

$$V(x) = -\frac{1}{\cosh^2 x}\left[\left(\frac{\alpha+\beta}{2}\right)^2 + \left(\frac{\alpha-\beta}{2}\right)^2 - \frac{1}{4}\right] + \frac{2\mathrm{i}\sinh x}{\cosh^2 x}\left(\frac{\beta+\alpha}{2}\right)\left(\frac{\beta-\alpha}{2}\right) \qquad (1)$$

as a typical illustration of the generic relations between the concept of $\mathcal{PT}$ symmetry and certain SUSYQM constructions. The Scarf II potential seems to be an ideal example for such a purpose since

- it is one of the shape-invariant potentials [25] which, in the SUSYQM context, belongs to type A factorization [4];
- its general functional form contains its Hermitian version (for $\alpha = \beta^*$ [26, 27]) as well as its $\mathcal{PT}$ symmetric one (for α and β both real or imaginary [24, 28]);
- in contrast to many $\mathcal{PT}$ symmetric potentials generated from the singular Hermitian potentials, its analysis [15, 23] does not require any artificial regularization;
- the spontaneous breakdown of its $\mathcal{PT}$ symmetry [9, 28–31] occurs simply due to a change in one of its real parameters to an imaginary value.

Before addressing the details, we note that our notation can be easily transformed to that used in other works. Thus, we might put $V_1 = [(\alpha+\beta)^2 + (\alpha-\beta)^2 - 1]/4$ and $V_2 = (\alpha+\beta)(\alpha-\beta)/2$ [29, 9], $A = -(\alpha+\beta+1)/2$ and $B = (\alpha-\beta)/2$ [30] and $s = -(\alpha+\beta+1)/2$ and $\lambda = \mathrm{i}(\alpha-\beta)/2$ [26, 27].

2.1. The Hermitian Scarf II potential

The conventional Hermitian version of the Scarf II potential [26, 27] is obtained when the second term in (1) is made real by the $\alpha = \beta^* = -s - \frac{1}{2} - \mathrm{i}\lambda$ parametrization, for example. One then finds the bound-state (normalizable) solutions at the energies

$$E_n = -\left(n + \frac{\alpha+\beta+1}{2}\right)^2 \qquad (2)$$

with the corresponding wavefunctions expressed in terms of Jacobi polynomials as

$$\psi_n(x) = C_n(1 - \mathrm{i}\sinh x)^{\frac{\alpha}{2}+\frac{1}{4}}(1 + \mathrm{i}\sinh x)^{\frac{\beta}{2}+\frac{1}{4}} P_n^{(\alpha,\beta)}(\mathrm{i}\sinh x). \qquad (3)$$

We note that although the Scarf II (or Gendenshtein) potential has been known for some time, the normalization constant C_n was determined only recently in [31]. The condition of the normalizability of the functions (3) limits the range of the admissible quantum numbers via

$$n < -[\mathrm{Re}(\alpha + \beta) + 1]/2. \tag{4}$$

We may note that the $\alpha \leftrightarrow \beta$ transformation changes the sign of the odd component of (1) and leaves the even one invariant. This mimics the spatial reflection operation $\mathcal{P}$ and has no effect on the energy spectrum. A simple calculation based on the properties of Jacobi polynomials reveals that the interchange $\alpha \leftrightarrow \beta$ acts as a spatial reflection on the wavefunctions (3), up to an unimportant sign change $(-1)^n$ in their norm [31].

We note that equations (1)–(4) apply to the general complex version of the Scarf II potential also, although in this case further refining the formalism might become necessary, as we shall see in the next subsection.

2.2. The $\mathcal{PT}$ symmetric Scarf II potential and the quasi-parity

In the light of the review [24], the $\mathcal{PT}$ symmetric version of the Scarf II potential is obtained if $\alpha^* = \pm\alpha$ and $\beta^* = \pm\beta$ holds, i.e. if α and β are both real or imaginary. The energy eigenvalues are all real, and the $\mathcal{PT}$ symmetry is unbroken, if both α and β are real. When one of the two parameters is real and the other is imaginary, the energy eigenvalues appear in complex conjugate pairs, and this case corresponds to the spontaneous breakdown of $\mathcal{PT}$ symmetry [28, 31]. If both parameters are imaginary, then there are no normalizable states due to the constraint (4).

Based on the practical equivalence of α and β, we can assume without any loss of generality that β is real and α is either real or imaginary, depending on whether we study unbroken or spontaneously broken $\mathcal{PT}$ symmetry. A remarkable feature of $\mathcal{PT}$ symmetric potentials is that the range of their normalizable states is broader than that of their Hermitian counterparts. In the case of the Scarf II potential, this is reflected by the fact that both α and $-\alpha$ can appear in equations (3), (2) and (4), since the potential (1) is invariant under the $\alpha \rightarrow -\alpha$ transformation.

The dual admissible sign of α may be called a quasi-parity quantum number [22, 30, 32] $q = \pm 1$. This makes the Scarf II potential similar to other $\mathcal{PT}$ symmetric potentials, which also have a second set of bound-state solutions compared to their Hermitian versions. However, the mechanism of the appearance of the second set is different from the scenario typical for the singular potentials where the singularity is cancelled by the $\mathcal{PT}$ symmetric regularization procedure. Now the new states 'evolve' from states that already existed as resonances in the Hermitian limit [10], so that their emergence is not directly related to the less strict boundary conditions.

In what follows we shall modify our notation slightly and replace α with $q\alpha$, where $q = \pm 1$ is the quasi-parity. This implies a redefinition of the formulae used previously as

$$E_n^{(q)} = -\left(n + \frac{q\alpha + \beta + 1}{2}\right)^2 \tag{5}$$

$$\psi_n^{(q\alpha,\beta)}(x) = C_n^{(q)}(1 - \mathrm{i}\sinh x)^{\frac{q\alpha}{2}+\frac{1}{4}}(1 + \mathrm{i}\sinh x)^{\frac{\beta}{2}+\frac{1}{4}} P_n^{(q\alpha,\beta)}(\mathrm{i}\sinh x). \tag{6}$$

The number of bound states contained in the two sets depends on q because the condition for normalizability is also modified accordingly,

$$n < -[\mathrm{Re}(q\alpha + \beta) + 1]/2. \tag{7}$$

In particular, when both α and β are real, i.e. the $\mathcal{PT}$ symmetry is unbroken, the bounds (7) differ in general; however, in the case of spontaneously broken $\mathcal{PT}$ symmetry, when α is

imaginary, the two sets contain equal numbers of bound states, as expected from the fact that the two sets are formed by complex conjugate energy eigenvalues.

In [31] the normalization constants in (6) have been determined using the modified inner product $\langle \psi | \mathcal{P} | \psi \rangle$ of [33]. The functions (6) were found to be orthogonal to each other using this inner product, while (as expected for a non-Hermitian problem), this was not the case when the standard Hermitian inner product was used.

3. Supersymmetrization of the $\mathcal{PT}$ invariant Scarf II potential

The duality of normalizable solutions in the $\mathcal{PT}$ symmetric setting implies a duality in the superpotentials also, and this is a remarkable new feature of the supersymmetrization of $\mathcal{PT}$ symmetric potentials. This means that in the realization of the standard $N = 2$ SUSYQM algebra [3]

$$\{Q, Q^\dagger\} = 0 \qquad \{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0 \qquad [\mathcal{H}, Q] = [\mathcal{H}, Q^\dagger] = 0 \quad (8)$$

the supersymmetric charge operators

$$Q = \begin{pmatrix} 0 & 0 \\ A^{(q)} & 0 \end{pmatrix} \qquad Q^\dagger = \begin{pmatrix} 0 & A^{\dagger(q)} \\ 0 & 0 \end{pmatrix} \qquad (9)$$

and the supersymmetric Hamiltonian

$$\mathcal{H} = \begin{pmatrix} H_-^{(q)} & 0 \\ 0 & H_+^{(q)} \end{pmatrix} \equiv \begin{pmatrix} A^{\dagger(q)} A^{(q)} & 0 \\ 0 & A^{(q)} A^{\dagger(q)} \end{pmatrix} \qquad (10)$$

are constructed using the SUSYQM shift operators

$$A^{(q)} = \frac{\mathrm{d}}{\mathrm{d}x} + W^{(q)}(x) \qquad A^{\dagger(q)} = -\frac{\mathrm{d}}{\mathrm{d}x} + W^{(q)}(x) \qquad (11)$$

which now depend explicitly on the quasi-parity quantum number $q = \pm 1$:

$$\begin{aligned} W^{(q)}(x) &= -\frac{\mathrm{d}}{\mathrm{d}x} \ln \psi_{0,-}^{(q\alpha,\beta)}(x) \\ &= -\frac{1}{2}(q\alpha + \beta + 1) \tanh x - \frac{\mathrm{i}}{2}(\beta - q\alpha) \frac{1}{\cosh x} \end{aligned} \qquad (12)$$

where $\psi_{n,-}^{(q\alpha,\beta)}(x) = \psi_n^{(q\alpha,\beta)}(x)$, so $\psi_{0,-}^{(q\alpha,\beta)}(x)$ is the ground-state wavefunction of the 'bosonic' Hamiltonian $H_-^{(q)}$.

The 'bosonic' potential may be constructed by using the standard SUSYQM recipe,

$$U_-^{(q)}(x) = [W^{(q)}(x)]^2 - \frac{\mathrm{d}W^{(q)}}{\mathrm{d}x} \qquad (13)$$

$$= -\frac{1}{\cosh^2 x} \left[\left(\frac{q\alpha + \beta}{2} \right)^2 + \left(\frac{q\alpha - \beta}{2} \right)^2 - \frac{1}{4} \right]$$

$$+ \frac{2\mathrm{i} \sinh x}{\cosh^2 x} \left(\frac{\beta + q\alpha}{2} \right) \left(\frac{\beta - q\alpha}{2} \right) + \left(\frac{q\alpha + \beta + 1}{2} \right)^2. \qquad (14)$$

This expression gives the same potential (1) as before, except for a constant term, which is simply an energy shift securing zero ground-state energy. In order to get back the original potential (1) and energy eigenvalues (5), we have to shift the energy scale, *subtracting the*

q-dependent constant term from the potential (and the energy expression) obtained in the SUSYQM procedure:

$$V_-^{(q\alpha,\beta)}(x) \equiv U_-^{(q)}(x) - \left(\frac{q\alpha + \beta + 1}{2}\right)^2 = V(x) \tag{15}$$

$$E_{n,-}^{(q\alpha,\beta)} = E_n^{(q)}. \tag{16}$$

Note that $V_-^{(q\alpha,\beta)}(x) = V(x)$ does not depend on q, as we have established previously. In what follows, therefore, we shall introduce the notation $\boldsymbol{H}_-$ for the 'bosonic' Hamiltonian in which the relative energy shift has been applied. In addition, the SUSYQM partners of the $U_-^{(q)}(x)$ potentials,

$$U_+^{(q)}(x) = [W^{(q)}(x)]^2 + \frac{\mathrm{d}W^{(q)}}{\mathrm{d}x} \tag{17}$$

contain the same energy constant as the $U_-^{(q)}(x)$ potentials (14). Subtracting these constants we get *two separate* potentials, which depend on q:

$$
\begin{aligned}
V_+^{(q\alpha,\beta)}(x) &\equiv U_+^{(q)}(x) - \left(\frac{q\alpha + \beta + 1}{2}\right)^2 \\
&= -\frac{1}{\cosh^2 x}\left[\left(\frac{q\alpha + \beta + 2}{2}\right)^2 + \left(\frac{q\alpha - \beta}{2}\right)^2 - \frac{1}{4}\right] \\
&\quad + \frac{2\mathrm{i}\sinh x}{\cosh^2 x}\left(\frac{\beta + q\alpha + 2}{2}\right)\left(\frac{\beta - q\alpha}{2}\right) = V_-^{(q\alpha+1,\beta+1)}(x).
\end{aligned} \tag{18}
$$

The main meaning of this formula is that *it assigns two different* supersymmetric partners to our original Scarf II potential and the corresponding Hamiltonian $\boldsymbol{H}_-$. We denote them by $\boldsymbol{H}_+^{(q)}$. Both potentials depend on the quasi-parity q and both of them have the shape of the Scarf II potential, with the parameters $q\alpha$ and β shifted by one unit. This is one of our main observations which extends the concept of shape invariance [25] to $\mathcal{PT}$ symmetric potentials.

The actual effect of the SUSYQM shift operators on the 'bosonic' and 'fermionic' eigenfunctions can be proved by straightforward but tedious calculations:

$$A^{(q)}\psi_{n,-}^{(q\alpha,\beta)}(x) = A^{(q)}\psi_n^{(q\alpha,\beta)}(x) \rightarrow \psi_{n-1}^{(q\alpha+1,\beta+1)}(x) = \psi_{n-1,+}^{(q\alpha,\beta)}(x) \tag{19}$$

$$A^{\dagger(q)}\psi_{n-1,+}^{(q\alpha,\beta)}(x) = A^{\dagger(q)}\psi_{n-1}^{(q\alpha+1,\beta+1)}(x) \rightarrow \psi_n^{(q\alpha,\beta)}(x) = \psi_{n,-}^{(q\alpha,\beta)}(x). \tag{20}$$

According to (7) the two partner potentials have one less bound (normalizable) state than the original Scarf II potential (1), and a comparison with (5) leads to the standard SUSYQM result for the energy eigenvalues of the 'fermionic' sector:

$$E_{n,+}^{(q\alpha,\beta)} = -\left(\frac{q\alpha + \beta + 3}{2} + n\right)^2 = E_{n+1}^{(q)} \equiv E_{n+1,-}^{(q\alpha,\beta)}. \tag{21}$$

Making use of the richer combination of SUSY shift operators and wavefunctions, we can analyse the effect of the $A^{(q)}$ operator on the 'bosonic' eigenfunctions with opposite quasi-parity $-q$:

$$A^{(q)}\psi_{n,-}^{(-q\alpha,\beta)}(x) = A^{(q)}\psi_n^{(-q\alpha,\beta)}(x) \rightarrow \psi_n^{(-q\alpha-1,\beta+1)}(x) = \psi_{n,+}^{(-q\alpha,\beta)}(x). \tag{22}$$

This result indicates that the 'fermionic' potential $V_+^{(-q\alpha,\beta)}(x) = V_-^{(-q(\alpha+1),\beta+1)}(x) = V_-^{(q(\alpha+1),\beta+1)}(x)$ has the same number of states with $-q$ as the original bosonic potential, and this is confirmed by a comparison between their spectra:

$$E_{n,+}^{(-q\alpha,\beta)} = -\left(\frac{-q\alpha + \beta + 1}{2} + n\right)^2 = E_n^{(-q)} \equiv E_{n,-}^{(-(q\alpha+1),\beta+1)}. \tag{23}$$
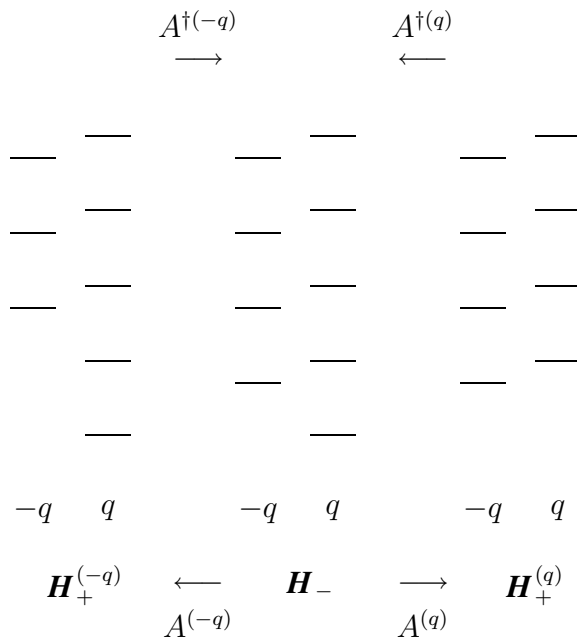
$$A^{\dagger(-q)} \qquad\qquad A^{\dagger(q)}$$
$$\longrightarrow \qquad\qquad \longleftarrow$$

$$-q \quad q \qquad\qquad -q \quad q \qquad\qquad -q \quad q$$

$$\boldsymbol{H}_+^{(-q)} \qquad \longleftarrow \qquad \boldsymbol{H}_- \qquad \longrightarrow \qquad \boldsymbol{H}_+^{(q)}$$
$$A^{(-q)} \qquad\qquad\qquad A^{(q)}$$

Figure 1. Schematic illustration of the relation between the spectra of the 'bosonic' Hamiltonian $\boldsymbol{H}_-$ and its two 'fermionic' partners $\boldsymbol{H}_+^{(q)}$ and $\boldsymbol{H}_+^{(-q)}$. In all three spectra levels with quasi-parity q $(-q)$ are degenerate with the corresponding levels in the other potentials, except that the lowest level with q $(-q)$ is missing from the spectrum of the 'fermionic' Hamiltonian $\boldsymbol{H}_+^{(q)}$ $(\boldsymbol{H}_+^{(-q)})$. The levels of $\boldsymbol{H}_+^{(q)}$ $(\boldsymbol{H}_+^{(-q)})$ are connected with those of the 'bosonic' Hamiltonian $\boldsymbol{H}_-$ by the $A^{(q)}$ and $A^{\dagger(q)}$ $(A^{(-q)}$ and $A^{\dagger(-q)})$ SUSY shift operators. The energy scale and the relative spacing of the energy levels are arbitrary.

The inverse operation of (22) is

$$A^{\dagger(q)}\psi_{n,+}^{(-q\alpha,\beta)}(x) = A^{\dagger(q)}\psi_n^{(-q(\alpha+1),\beta+1)}(x) \rightarrow \psi_n^{(-q\alpha,\beta)}(x) = \psi_{n,-}^{(-q\alpha,\beta)}(x). \qquad (24)$$

The situation is schematically illustrated in figure 1.

The results in (22) and (24) are analogous to some relations found in a SUSYQM inspired study of the $\mathcal{PT}$ symmetric spiked harmonic oscillator [18], and they are practically equivalent to the relations describing the effect of the two sets of $so(2, 1)$ generators on the two sets of solutions of the $\mathcal{PT}$ symmetric Scarf II potential in the algebraic analysis [11]. We also note that a similar system of partner potentials has been obtained [34] from two essentially different supersymmetric constructions; i.e. the parasupersymmetric scheme (where a three- rather than a two-dimensional matrix representation is used in (9) and (10)) and second-order supersymmetry (where A and $A^\dagger$ in (11) are second- rather than first-order differential operators).

It has to be added though that the new degeneracy patterns and partnerships of potentials are not specific to the Scarf II potential, rather they are valid for any system where supersymmetry and $\mathcal{PT}$ symmetry appear simultaneously. To show this, we assume that similar to the situation for the Scarf II potential, the 'bosonic' Hamiltonian $\boldsymbol{H}_-$ can be made independent of the quasi-parity quantum number by applying an appropriate relative energy shift of the $q = +1$ and $q = -1$ sectors, i.e. $\boldsymbol{H}_- = A^{\dagger(q)}A^{(q)} + \varepsilon^{(q)} = A^{\dagger(-q)}A^{(-q)} + \varepsilon^{(-q)}$.

With this assumption the isospectrality of the two 'fermionic' Hamiltonians follows from the eigenvalue equations

$$\boldsymbol{H}_{-}\psi_{n,-}^{(q)} = [A^{\dagger(\pm q)}A^{(\pm q)} + \varepsilon^{(\pm q)}]\psi_{n,-}^{(q)} = E_{n,-}^{(q)}\psi_{n,-}^{(q)} \tag{25}$$

$$\boldsymbol{H}_{+}^{(\pm q)}\psi_{n,+}^{(q)} = [A^{(\pm q)}A^{\dagger(\pm q)} + \varepsilon^{(\pm q)}]\psi_{n,+}^{(q)} = E_{n,+}^{(q)}\psi_{n,+}^{(q)}. \tag{26}$$

From (25) it follows that the energy shift is related to the 'bosonic' ground-state energy as $\varepsilon^{(\pm q)} = E_{0,-}^{(\pm q)}$, as indeed was the case for the Scarf II potential (see (5), (15) and (16)). With the above eigenvalue equations,

$$\begin{aligned}
A^{(q)}\boldsymbol{H}_{-}\psi_{n,-}^{(q)} &= A^{(q)}[A^{\dagger(q)}A^{(q)} + \varepsilon^{(q)}]\psi_{n,-}^{(q)} \\
&= \boldsymbol{H}_{+}^{(q)}A^{(q)}\psi_{n,-}^{(q)} \\
&= E_{n,-}^{(q)}A^{(q)}\psi_{n,-}^{(q)}
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
A^{(-q)}\boldsymbol{H}_{-}\psi_{n,-}^{(q)} &= A^{(-q)}[A^{\dagger(-q)}A^{(-q)} + \varepsilon^{(-q)}]\psi_{n,-}^{(q)} \\
&= \boldsymbol{H}_{+}^{(-q)}A^{(-q)}\psi_{n,-}^{(q)} \\
&= E_{n,-}^{(q)}A^{(-q)}\psi_{n,-}^{(q)}
\end{aligned} \tag{28}$$

so we find that the $A^{(\pm q)}\psi_{n,-}^{(q)}$ functions are eigenfunctions of the $\boldsymbol{H}_{+}^{(\pm q)}$ 'fermionic' Hamiltonians, and the corresponding energy eigenvalues are the same as those of the q-independent 'bosonic' Hamiltonian. There is a difference between (27) and (28) that in the former case $A^{(q)}\psi_{n,-}^{(q)} = 0$ by construction, so the partner of the ground-state 'bosonic' level is missing from the 'fermionic' Hamiltonian $\boldsymbol{H}_{+}^{(q)}$, while the situation is different for (28), so there the number of levels is the same in the 'bosonic' and 'fermionic' Hamiltonians, just as we have seen for the example of the Scarf II potential. These considerations prove that the combination of the two symmetries leads to a richer spectral pattern than either of them separately, not only for the Scarf II potential, but also for any q-independent potentials.

4. Discussion

After one moves from the analytic to the algebraic context, the doubling of solutions may be seen as directly reflecting the fact that the $su(1, 1) \sim so(2, 1)$ algebra associated with the solutions [10] becomes doubled and eventually leads to a larger $so(2, 2)$ potential algebra of the model [11]. In fact, the doubling of the states is one of the key motivations of our present analysis. It indicates that instead of one SUSY partner, $\mathcal{PT}$ symmetric potentials can have two, i.e. one with $q = 1$ and another with $q = -1$. Similar results have been obtained for the $\mathcal{PT}$ symmetrized spiked harmonic oscillator in [18]; however, the visualization and transparency of the example were significantly obscured there by the presence of the essential singularity of the solutions at $x = 0$ [35]. The smooth Scarf II potential is much better suited for similar purposes.

We found that the two 'fermionic' partner Hamiltonians are isospectral with the 'bosonic' one in the sense that the spectrum of $\boldsymbol{H}_{+}^{(q)}$ ($\boldsymbol{H}_{+}^{(-q)}$) misses the level corresponding to $E_{0,-}^{(q)}$ ($E_{0,-}^{(-q)}$) level of the 'bosonic' Hamiltonian $\boldsymbol{H}_{-}$. This isospectrality has been demonstrated explicitly in the case of the Scarf II potential, but we also showed that it holds for any potential characterized simultaneously by $\mathcal{PT}$ symmetry and supersymmetry. The situation is illustrated schematically in figure 1. For the Scarf II potential this network of interrelated levels is practically the same as the one obtained in terms of an $so(2, 2)$ potential algebra [11],

and the two sets of SUSY shift operators correspond to the two sets of $so(2, 1)$ ladder operators. It has to be stressed that although the spectra of Hamiltonians with opposite quasi-parity are interrelated here, there is no operator which would flip the q of a given wavefunction, so the two quasi-parity sectors remain disjoint in this sense.

The results obtained for the Scarf II potential have significantly different implications for unbroken and broken $\mathcal{PT}$ symmetry. In the former case the 'fermionic' partner potentials (18) are $\mathcal{PT}$ symmetric, and the $E_{n,+}^{(q\alpha,\beta)}$ energy eigenvalues remain real. In the latter case, however, the coupling parameters of both the even and the odd components of the potential become complex due to the imaginary value of α, therefore the 'fermionic' potentials cease to be $\mathcal{PT}$ symmetric. This manifest breakdown of $\mathcal{PT}$ symmetry is also demonstrated by the fact that the complex energy eigenvalues cease to appear in complex conjugated pairs in the spectrum of the 'fermionic' Hamiltonians, because the equivalent of the $E_{0,-}^{(\pm q\alpha,\beta)}$ 'bosonic' state will be missing from the spectrum of the $H_+^{(\pm q)}$ 'fermionic' Hamiltonian.

We note that a generalized form of the $\mathcal{PT}$ invariant Scarf II potential can be obtained by an *imaginary* shift of the axis of coordinates, $x \to x + \mathrm{i}\epsilon$ [24, 36]. Such a transformation cannot influence any of the conclusions of our work since the modified potential and eigenfunctions behave in the same way under the $\mathcal{PT}$ transformation as the original ones, only their functional form becomes more complicated when we decide to stay on the real line and change the variables accordingly. This clarifies why the energy eigenvalues are independent of ϵ as well as why the change may only influence the wavefunctions and/or the reflection and transmission coefficients [10].

Turning now to more general considerations, we note that in the $\mathcal{PT}$ symmetric setting the supersymmetrization can be realized in an alternative way, as discussed in [12]. In such a framework the SUSY algebra is to be realized by the operators that contain the antilinear $\mathcal{T}$ operation explicitly. One need not even use any particular potential to reveal the relation of this scheme to the conventional one [37]. It suffices to recollect that the SUSY charge and shift operators may contain the time reflection (i.e., complex conjugation) operator $\mathcal{T}$, say, in the form

$$\widetilde{Q} = \begin{pmatrix} 0 & 0 \\ \mathcal{T}A^{(q)} & 0 \end{pmatrix} \qquad \widetilde{Q}^\dagger = \begin{pmatrix} 0 & A^{\dagger(q)}\mathcal{T} \\ 0 & 0 \end{pmatrix}. \tag{29}$$

Consequently, the SUSY Hamiltonian is different in its 'fermionic' component

$$\widetilde{\mathcal{H}} = \begin{pmatrix} \widetilde{H}_-^{(q)} & 0 \\ 0 & \widetilde{H}_+^{(q)} \end{pmatrix} \equiv \begin{pmatrix} A^{\dagger(q)}A^{(q)} & 0 \\ 0 & \mathcal{T}A^{(q)}A^{\dagger(q)}\mathcal{T} \end{pmatrix}. \tag{30}$$

This indicates that the 'bosonic' component of the modified Hamiltonian is the same as in the original case (10), $\widetilde{H}_-^{(q)} = H_-^{(q)}$, while the 'fermionic' component of the modified Hamiltonian coincides with the complex conjugate of the original 'fermionic' Hamiltonian $\widetilde{H}_+^{(q)} = \mathcal{T}H_+^{(q)}\mathcal{T}$. Introducing the shifted energy scale as in (25) and (26), these relations become $\widetilde{H}_- = H_-$ and $\widetilde{H}_+^{(q)} = \mathcal{T}H_+^{(q)}\mathcal{T} + [\varepsilon^{(q)}]^*$. For unbroken $\mathcal{PT}$ symmetry of H_-, i.e. when the energy eigenvalues are real and consequently $\varepsilon^{(q)}$ is also real, this means that the energy eigenvalues of $\widetilde{H}_+^{(q)}$ are also real, while for spontaneously broken $\mathcal{PT}$ symmetry, when the energy eigenvalues and $\varepsilon^{(q)}$ are complex, the energy eigenvalues of $\widetilde{H}_+^{(q)}$ are the complex conjugates of the eigenvalues of $H_+^{(q)}$. The eigenfunctions are equally trivially related to the original 'fermionic' eigenfunctions in both cases.

Furthermore, the $\mathcal{PT}$ invariance leads to a special relation between the $\mathcal{P}$ and $\mathcal{T}$ operations themselves. If $H_+^{(q)}$ is $\mathcal{PT}$ symmetric, then the complex conjugation operation has the same effect on it as the $\mathcal{P}$ spatial reflection operation, so $\widetilde{H}_+^{(q)}$ contains the spatially reflected potential

appearing in $\boldsymbol{H}_+^{(q)}$, so the modified SUSY construction does not differ essentially from the usual one. A similar relation holds between the eigenfunctions, if they are eigenfunctions of the $\mathcal{PT}$ operator, i.e. if the $\mathcal{PT}$ symmetry is unbroken. The energy eigenvalues of $\widetilde{\boldsymbol{H}}_+^{(q)}$ are real and the same as those of $\boldsymbol{H}_+^{(q)}$, as we have seen above. In the case of spontaneously broken $\mathcal{PT}$ symmetry, the situation is different since the eigenfunctions are no longer invariant under the $\mathcal{PT}$ operation. The energy eigenvalues remain the same since the complex conjugate pairs simply transform into themselves under complex conjugation. However, in the case of the spontaneously broken $\mathcal{PT}$ symmetry, the $\mathcal{PT}$ invariance of $\widetilde{\boldsymbol{H}}_- = \boldsymbol{H}_-^{(q)}$ need not lead to the $\mathcal{PT}$ invariance of $\widetilde{\boldsymbol{H}}_+^{(q)}$ (and thus to that of $\mathcal{T}U_+^{(q)}(x)\mathcal{T} = U_+^{(-q)}(x)$, as we have seen in the example of the Scarf II potential), so the whole SUSY construction can break down in this case.

5. Summary

In general, solvable $\mathcal{PT}$ symmetric potentials may have a richer spectrum than their Hermitian counterparts. *A priori*, this feature may have non-trivial implications in the SUSY constructions. We investigated the $\mathcal{PT}$ symmetric version of the Scarf II potential in the role of a 'bosonic' potential and described in detail the construction of its 'fermionic' SUSY partners.

Our first finding was based on the familiar knowledge that our potential possesses two sets of normalizable eigenfunctions distinguished by their quasi-parity quantum number $q = \pm 1$. On this basis we arrived at our first important observation that one can introduce *two different* superpotential functions which lead to *two different* SUSY partners of the original potential.

The formal application of the standard rules of supersymmetric quantum mechanics requires the vanishing ground-state energy of the 'bosonic' potential, so a relative energy shift of the $q = 1$ and $q = -1$ sectors is needed to correlate the energy scales. (This corresponds to switching to the Hamiltonians $\boldsymbol{H}_-$ and $\boldsymbol{H}_+^{(q)}$ instead of $H_-^{(q)}$ and $H_+^{(q)}$.) With this shift the 'bosonic' potential can be made independent of the quasi-parity quantum number q. We found that the two partner potentials are Scarf II potentials with the parameters $\pm q\alpha + 1$ and $\beta + 1$. Their energy spectrum contains one less normalizable state than the 'bosonic' potential, and the missing level carries the same quasi-parity quantum number as the 'fermionic' Hamiltonian $\boldsymbol{H}_+^{(q)}$.

This is similar to the structures found within the standard SUSYQM, although the related concept of the shape invariance must be modified slightly. Still, the situation becomes perceivably different for the unbroken and for the spontaneously broken $\mathcal{PT}$ symmetry. In the former case, both α and β and the energy eigenvalues are real. In this case both 'fermionic' partner potentials exhibit $\mathcal{PT}$ symmetry. In contrast, for spontaneously broken $\mathcal{PT}$ symmetry (i.e. when α is imaginary, β is real and the energy eigenvalues form complex conjugate pairs) the $\mathcal{PT}$ symmetry of the 'fermionic' partner potentials may become broken *manifestly*. Our second important observation is that the isospectrality of the potentials (with the exception of the 'bosonic' $n = 0$ states) still holds in this latter case, too.

Additionally, we considered an alternative SUSY construction for $\mathcal{PT}$ symmetric potentials in the spirit of [12] where the SUSY algebra has been realized by SUSY charge operators containing the antilinear operator $\mathcal{T}$ in an explicit form. Our considerations were not restricted to a particular potential, rather their validity was universal. Again, the 'bosonic' Hamiltonian was the same as in the standard SUSYQM approach to $\mathcal{PT}$ symmetric potentials, while the 'fermionic' partner Hamiltonians (with $q = \pm 1$) proved to be the complex conjugates of the original 'fermionic' Hamiltonians. If the 'fermionic' potentials possessed $\mathcal{PT}$ symmetry

(as was the case for the 'bosonic' Scarf II potential with unbroken $\mathcal{PT}$ symmetry), then the $\mathcal{T}$ operation had the same effect on them as $\mathcal{P}$. They simply proved to be the spatially reflected versions of the 'fermionic' partner potentials in the standard SUSYQM setting. Their spectrum was the same and, therefore, the wavefunctions were related to the original 'fermionic' wavefunctions in a trivial way. It has to be emphasized, nevertheless, that the isospectrality with the original 'fermionic' potential holds even when the 'fermionic' potentials are not $\mathcal{PT}$ symmetric (as was the case for the 'bosonic' Scarf II potential with spontaneously broken $\mathcal{PT}$ symmetry).

Acknowledgments

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